Quantum Root Vectors and a Dolbeault Double Complex for the A-Series Quantum Flag Manifolds

Réamonn Ó Buachalla

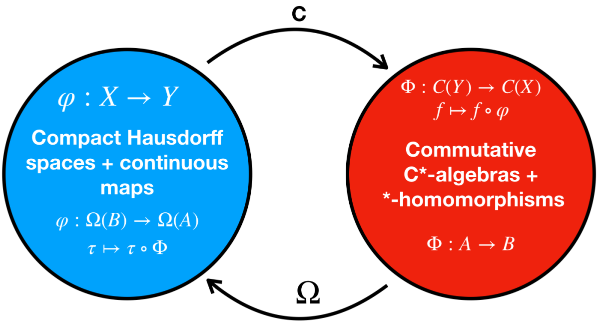
Mathematical Institute, Charles University in Prague

Noncommutative Geometry and Topology Seminar

(Joint work with Petr Somberg)

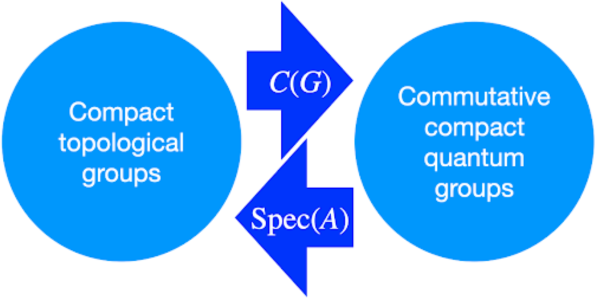
0: Some C^*-Motivation

Recall the duality of categories:

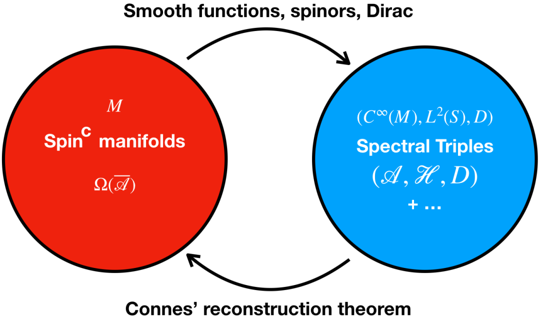


$\Omega(A) := \{\tau : A \to \mathbb{C} \mid \tau$ **a *-homomorphism**$\}$,
equipped with weak-* topology

Woronowicz extended Gelfand duality to a “topological group duality”.



Connes' Reconstruction Theorem extends Gelfand duality to a "differential duality".



1: Drinfeld–Jimbo Quantum Groups

- Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and G the associated compact connected simply connected Lie group.

1: Drinfeld–Jimbo Quantum Groups

- Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and G the associated compact connected simply connected Lie group.
- Emerging from mathematical physics in the 1980s came a dual pairing of Hopf algebras:

$$U_q(\mathfrak{g}) \times \mathcal{O}_q(G) \to \mathbb{C},$$

1: Drinfeld–Jimbo Quantum Groups

- Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and G the associated compact connected simply connected Lie group.
- Emerging from mathematical physics in the 1980s came a dual pairing of Hopf algebras:

$$U_q(\mathfrak{g}) \times \mathcal{O}_q(G) \to \mathbb{C},$$

where as $q \to 1$,

$$\mathcal{O}_q(G) \to \mathcal{O}(G),$$

and $U_q(\mathfrak{g})$ goes to a $(\text{rank}(\mathfrak{g}) + 1)$-fold cover of $U(\mathfrak{g})$.

For $q \in \mathbb{R}_{>0}$, the deformation $U_q(\mathfrak{g})$ has many important properties:

For $q \in \mathbb{R}_{>0}$, the deformation $U_q(\mathfrak{g})$ has many important properties:

- Its category of finite-dimensional representations ${}_{U_q(\mathfrak{g})}\mathrm{mod}$ is equivalent to ${}_{U(\mathfrak{g})}\mathrm{mod}$ the category of finite-dimensional representations of $U(\mathfrak{g})$.

For $q \in \mathbb{R}_{>0}$, the deformation $U_q(\mathfrak{g})$ has many important properties:

- Its category of finite-dimensional representations ${}_{U_q(\mathfrak{g})}\mathrm{mod}$ is equivalent to ${}_{U(\mathfrak{g})}\mathrm{mod}$ the category of finite-dimensional representations of $U(\mathfrak{g})$.
- It has a Hopf algebra structure, but the associated monoidal structure on ${}_{U_q(\mathfrak{g})}\mathrm{mod}$ is **not** monoidally equivalent to the standard monoidal structure of ${}_{U(\mathfrak{g})}\mathrm{mod}$.

For $q \in \mathbb{R}_{>0}$, the deformation $U_q(\mathfrak{g})$ has many important properties:

- Its category of finite-dimensional representations ${}_{U_q(\mathfrak{g})}\mathrm{mod}$ is equivalent to ${}_{U(\mathfrak{g})}\mathrm{mod}$ the category of finite-dimensional representations of $U(\mathfrak{g})$.
- It has a Hopf algebra structure, but the associated monoidal structure on ${}_{U_q(\mathfrak{g})}\mathrm{mod}$ is **not** monoidally equivalent to the standard monoidal structure of ${}_{U(\mathfrak{g})}\mathrm{mod}$.
- In a sense which can be made precise, this is the unique q-deformation of the monoidal structure of ${}_{U(\mathfrak{g})}\mathrm{mod}$. Moreover, it comes endowed with a unique braiding.

- As we now understand quite well, the classical topology of each G admits a direct q-deformation expressible in Woronowicz's C^*-algebraic framework of *compact quantum groups*.

- As we now understand quite well, the classical topology of each G admits a direct q-deformation expressible in Woronowicz's C^*-algebraic framework of *compact quantum groups*.

Question

Does the classical differential geometry of G admit an analogous q-deformation?

- As we now understand quite well, the classical topology of each G admits a direct q-deformation expressible in Woronowicz's C^*-algebraic framework of *compact quantum groups*.

Question

Does the classical differential geometry of G admit an analogous q-deformation?

- We also now understand that

- As we now understand quite well, the classical topology of each G admits a direct q-deformation expressible in Woronowicz's C^*-algebraic framework of *compact quantum groups*.

Question

Does the classical differential geometry of G admit an analogous q-deformation?

- We also now understand that this is a much more difficult question!

2: Differential Calculi

- Where to start looking for such a q-deformed geometry?

2: Differential Calculi

- Where to start looking for such a q-deformed geometry?
- Woronowicz's idea was to look for q-deformations of the de Rham complex.

Definition

A pair $(\Omega^\bullet, d)$ is called a **differential graded algebra** if $\Omega^\bullet = \bigoplus_{k\in\mathbf{N}_0} \Omega^k$ is an $\mathbf{N}_0$-graded algebra, and d is a degree 1 map such that $d^2 = 0$, and

$$\mathrm{d}(\omega \wedge \nu) = \mathrm{d}(\omega) \wedge \nu + (-1)^k \omega \wedge \mathrm{d}(\nu), \qquad (\omega \in \Omega^k, \nu \in \Omega^\bullet).$$

Definition

A **differential calculus** is a differential graded algebra $(\Omega^\bullet, \mathrm{d})$ generated in degree-0,

Definition

A **differential calculus** is a differential graded algebra $(\Omega^\bullet, \mathrm{d})$ generated in degree-0, i.e. generated by those elements of the form a and $\mathrm{d}b$, for $a, b \in A$.

Definition

A **differential calculus** is a differential graded algebra $(\Omega^\bullet, \mathrm{d})$ generated in degree-0, i.e. generated by those elements of the form a and $\mathrm{d}b$, for $a, b \in A$.

We say that $(\Omega^\bullet, \mathrm{d})$ is **left covariant** if it admits a left $U_q(\mathfrak{g})$-module algebra structure, with respect to which d is $U_q(\mathfrak{g})$-module map. Similarly we define **right and bicovariant** calculi.

Problem

There do not exist any bicovariant calculi over $\mathcal{O}_q(G)$ of classical dimension!

3: Quantum Flag Manifolds and the Heckenberger–Kolb Calculi

- The dual pairing $U_q(\mathfrak{g}) \times \mathcal{O}_q(G) \to \mathbb{C}$ gives an action

$$U_q(\mathfrak{g}) \times \mathcal{O}_q(G) \to \mathcal{O}_q(G).$$

3: Quantum Flag Manifolds and the Heckenberger–Kolb Calculi

- The dual pairing $U_q(\mathfrak{g}) \times \mathcal{O}_q(G) \to \mathbb{C}$ gives an action

$$U_q(\mathfrak{g}) \times \mathcal{O}_q(G) \to \mathcal{O}_q(G).$$

- In particular, we have an action

$$U_q(\mathfrak{sl}_2) \times \mathcal{O}_q(SU_2) \to \mathcal{O}_q(SU_2),$$

3: Quantum Flag Manifolds and the Heckenberger–Kolb Calculi

- The dual pairing $U_q(\mathfrak{g}) \times \mathcal{O}_q(G) \to \mathbb{C}$ gives an action

$$U_q(\mathfrak{g}) \times \mathcal{O}_q(G) \to \mathcal{O}_q(G).$$

- In particular, we have an action

$$U_q(\mathfrak{sl}_2) \times \mathcal{O}_q(SU_2) \to \mathcal{O}_q(SU_2),$$

which gives an action

$$U_q(\mathfrak{h}) \times \mathcal{O}_q(SU_2) \to \mathcal{O}_q(SU_2),$$

- Classically, the invariants give

$$^{U_q(\mathfrak{h})}\mathcal{O}(SU_2) = \mathcal{O}(S^2).$$

Recall that we have an isomorphism

$$S^2 \simeq \mathbb{CP}^1 \simeq SU_2/U_1.$$

- In the quantum setting, the invariant space

$$^{U_q(\mathfrak{h})}\mathcal{O}_q(SU_2) = \mathcal{O}_q(S^2),$$

is called the *Podleś sphere*.

- Classically, the invariants give

$$^{U_q(\mathfrak{h})}\mathcal{O}(SU_2) = \mathcal{O}(S^2).$$

Recall that we have an isomorphism

$$S^2 \simeq \mathbb{CP}^1 \simeq SU_2/U_1.$$

- In the quantum setting, the invariant space

$$^{U_q(\mathfrak{h})}\mathcal{O}_q(SU_2) = \mathcal{O}_q(S^2),$$

is called the *Podleś sphere*.

- It admits a direct left $U_q(\mathfrak{sl}_2)$-covariant q-deformation of its de Rham complex, with an extremely rich and interesting noncommutative geometry!!

- The 2-sphere S^2 is a compact simply-connected SU_2-homogeneous Kähler manifold.
- In general, a compact simply-connected G-homogeneous Kähler manifold is called a *flag manifold*.
- They can equivalently be presented as quotients of the form

$$G/L_S$$

where L_S is a Levi subgroup,

- The 2-sphere S^2 is a compact simply-connected SU_2-homogeneous Kähler manifold.
- In general, a compact simply-connected G-homogeneous Kähler manifold is called a *flag manifold*.
- They can equivalently be presented as quotients of the form

$$G/L_S$$

where L_S is a Levi subgroup, roughly speaking L_S is a subgroup of G containing a maximal torus. They are indexed by subsets S of the simple roots of $\mathfrak{g}$.

- For S a subset of simple roots, we have the *quantum Levi subalgebra*

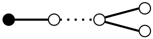
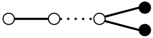
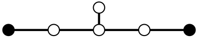
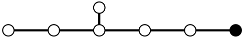
$$U_q(\mathfrak{l}_S) := \langle K_i, E_j, F_j \,|\, i = 1, \ldots, r; j \in S \rangle \subseteq U_q(\mathfrak{sl}_n).$$

Definition

For S a subset of simple roots of $\mathfrak{g}$, the corresponding *quantum flag manifold* is the invariant subspace

$$\begin{aligned}\mathcal{O}_q(G/L_S) &:= {}^{U_q(\mathfrak{l}_S)}\mathcal{O}_q(G) \\ &= \{ g \in \mathcal{O}_q(G) | X \triangleright g = \varepsilon(X) g, \forall X \in U_q(\mathfrak{l}_S) \}.\end{aligned}$$

Compact Quantum Hermitian Symmetric Spaces

A_n		$\mathcal{O}_q(\mathrm{Gr}_{n,r})$	quantum Grassmanian
B_n		$\mathcal{O}_q(\mathrm{Q}_{2n+1})$	odd quantum quadric
C_n		$\mathcal{O}_q(\mathrm{L}_n)$	symmetric q.-Lagrangian Grassmannian
D_n		$\mathcal{O}_q(\mathrm{Q}_{2n})$	even quantum quadric
D_n		$\mathcal{O}_q(\mathrm{S}_n)$	quantum spinor variety
E_6		$\mathcal{O}_q(\mathbb{OP}^2)$	quantum Cayley plane
E_7		$\mathcal{O}_q(\mathrm{F})$	quantum Freudenthal variety

- By construction, each quantum flag manifold $\mathcal{O}_q(G/L_S)$ is a right $U_q(\mathfrak{g})$-submodule of $\mathcal{O}_q(G)$, meaning it makes sense to talk about right covariant differential calculi.

- By construction, each quantum flag manifold $\mathcal{O}_q(G/L_S)$ is a right $U_q(\mathfrak{g})$-submodule of $\mathcal{O}_q(G)$, meaning it makes sense to talk about right covariant differential calculi.

Theorem (Heckenberger, Kolb '06)

For each quantum flag manifold $\mathcal{O}_q(G/L_S)$ of Hermitian symmetric type, there exists a unique right covariant differential calculus $\Omega^\bullet_q(G/L_S)$ of classical dimension.

- In the 15 years since these calculi were discovered, we have learned a lot about their structure: complex and Kähler geometry, cohomology, and their completions to spectral triples in the sense of Connes.

- In the 15 years since these calculi were discovered, we have learned a lot about their structure: complex and Kähler geometry, cohomology, and their completions to spectral triples in the sense of Connes.

Questions

1) Where do these differential calculi come from?

- In the 15 years since these calculi were discovered, we have learned a lot about their structure: complex and Kähler geometry, cohomology, and their completions to spectral triples in the sense of Connes.

Questions

1) Where do these differential calculi come from?
2) What about $\mathcal{O}_q(G/L_S)$ of non-Hermitian symmetric type?

- In the 15 years since these calculi were discovered, we have learned a lot about their structure: complex and Kähler geometry, cohomology, and their completions to spectral triples in the sense of Connes.

Questions

1) Where do these differential calculi come from?
2) What about $\mathcal{O}_q(G/L_S)$ of non-Hermitian symmetric type?
3) Can this approach be extended to $\mathcal{O}_q(G)$?

4: Non-Hermitian Symmetric Quantum Flags

- Left $\mathcal{O}(G)$-covariant differential calculi over quantum flag manifolds correspond to **tangent spaces** $T \subseteq U_q(\mathfrak{g})$, satisfying

$$T(1) = 0, \quad \Delta(T) \subseteq \mathcal{O}_q(G/L_S)^\circ \otimes T, \quad U_q(\mathfrak{l}_S)T \subseteq T.$$

4: Non-Hermitian Symmetric Quantum Flags

- Left $\mathcal{O}(G)$-covariant differential calculi over quantum flag manifolds correspond to **tangent spaces** $T \subseteq U_q(\mathfrak{g})$, satisfying

$$T(1) = 0, \quad \Delta(T) \subseteq \mathcal{O}_q(G/L_S)^\circ \otimes T, \quad U_q(\mathfrak{l}_S)T \subseteq T.$$

- More precisely, associated to each tangent space T, we have a space of **tangent vectors**

$$\chi := \mathcal{O}_q \square_{U_q(\mathfrak{l}_S)} T = \mathcal{O}_q \square_{\mathcal{O}_q(L_S)} T$$

and a dual **space of 1-forms**, i.e. a dg-algebra of length 1

$$\mathcal{O}_q(G/L_S) \xrightarrow{\mathrm{d}} \Omega^1_q(G/L_S) := \mathcal{O}_q(G) \square_{\mathcal{O}_q(L_S)} T^*.$$

- The $(\Omega^1_q(G/L_S), \mathrm{d})$ can then be extended to a dg-algebra $\Omega^\bullet_q(G/L_S), \mathrm{d})$ of maximal length, and it is universal with respect to this property.

Lesson

We can find and classify differential calculi by looking at quantum tangent spaces $T \subseteq U_q(\mathfrak{g})^\circ$.

- The $(\Omega^1_q(G/L_S), \mathrm{d})$ can then be extended to a dg-algebra $\Omega^\bullet_q(G/L_S), \mathrm{d})$ of maximal length, and it is universal with respect to this property.

Lesson

We can find and classify differential calculi by looking at quantum tangent spaces $T \subseteq U_q(\mathfrak{g})^\circ$.

- How to extend beyond the Hermitian symmetric situation?

- Classically we have an split exact sequence

$$0 \to \mathfrak{l}_S \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{l}_S \to 0,$$

which is to say a direct sum decomposition

$$\mathfrak{g} \simeq \mathfrak{l}_S \oplus T.$$

- Classically we have an split exact sequence

$$0 \to \mathfrak{l}_S \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{l}_S \to 0,$$

which is to say a direct sum decomposition

$$\mathfrak{g} \simeq \mathfrak{l}_S \oplus T.$$

- In the quantum setting there is no generally accepted "quantum Lie subalgebra"

$$\text{``}\mathfrak{g}_q\text{''} \hookrightarrow U_q(\mathfrak{g}).$$

Theorem (Braid group action)

To every i, $i = 1, \ldots, r$, there corresponds an algebra automorphism T_i of $U_q(\mathfrak{g})$ which acts on the generators as

$$T_i(K_j) = K_j K_i^{-a_{ij}}, \quad T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i,$$

$$T_i(E_j) = \sum_{t=0}^{-a_{ij}} (-1)^{t-a_{ij}} q_i^{-t} (E_i)^{(-a_{ij}-t)} E_j (E_i)^{(t)}, \quad i \neq j,$$

$$T_i(F_j) = \sum_{t=0}^{-a_{ij}} (-1)^{t-a_{ij}} q_i^{-t} (F_i)^{(t)} E_j (F_i)^{(-a_{ij}-t)}, \quad i \neq j,$$

Theorem

. . . where

$$(E_i)^{(n)} := E_i^n/[n]_q!, \quad (F_i)^{(n)} := F_i^n/[n]_q!.$$

The mapping $w_i \to T_i$ determines a homomorphism of the Braid group $\mathfrak{B}_{\mathfrak{g}}$ into the group of algebra automorphism of $U_q(\mathfrak{g})$.

Theorem

. . . where

$$(E_i)^{(n)} := E_i^n/[n]_q!, \quad (F_i)^{(n)} := F_i^n/[n]_q!.$$

The mapping $w_i \to T_i$ determines a homomorphism of the Braid group $\mathfrak{B}_{\mathfrak{g}}$ into the group of algebra automorphism of $U_q(\mathfrak{g})$.

Definition

For an element $w \in W$ with reduced decomposition $w = w_{i_1} \cdots w_{i_n}$ we define

$$E_{\beta_r} = T_{i_1} T_{i_2} \cdots T_{i_{r-1}}(E_{i_r}), \qquad F_{\beta_r} = T_{i_1} T_{i_2} \cdots T_{i_{r-1}}(F_{i_r})$$

and call them **root vectors** of $U_q(\mathfrak{g})$.

Theorem (Lusztig's PBW Basis)

The following set of elements is a vector basis of $U_q(\mathfrak{g})$:

$$F_{\beta_1}^{r_1} \cdots F_{\beta_n}^{r_n} K_1^{t_1} \cdots K_l^{t_l} E_{\beta_n}^{s_n} \cdots F_{\beta_1}^{r_1},$$

where $r_i, t_j, s_k \in \mathbb{Z}_{\geq 0}$.

- This result is the starting point for Luzstig's theory of canonical bases and much more . . .

Definition

For any choice of decomposition I of the longest element of the Weyl group, we denote by T_I the span of the root vectors

$$\{F_{\beta_1}, \ldots, F_{\beta_n}, E_{\beta_1}, \ldots, E_{\beta_n}\}.$$

Definition

The *full quantum flag manifold* $\mathcal{O}_q(F_{n+1})$ of $\mathcal{O}_q(G)$ is the quantum flag manifold where S consists of all nodes of the Dynkin diagram of $U_q(\mathfrak{g})$.

- It is important to note that an inclusion of sets $S' \subseteq S$ implies an inclusion of algebras $\mathcal{O}_q(G/L_{S'}) \subseteq \mathcal{O}_q(G/L_S)$. Hence **full quantum flags contain all other quantum flags.**

Theorem (RÓB, P. Somberg 2021)

For $U_q(\mathfrak{sl}_{n+1})$, precisely the following two decompositions

$$I := (w_1 w_2 \cdots w_n)(w_1 w_2 \cdots w_{n-1}) \cdots (w_1 w_2) w_1,$$
$$I' := (w_n w_{n-1} \cdots w_1)(w_n w_{n-1} \cdots w_2) \cdots (w_n w_{n-1}) w_n,$$

that T_I and $T_{I'}$ are quantum tangent spaces, for the full quantum flag manifold $\mathcal{O}_q(F_{n+1})$, with associated differential calculi $(\Omega_q^\bullet(F_{n+1}), \mathrm{d})$ of classical dimension.

Corollary

For both I and I′ the differential calculus $(\Omega_q^\bullet(F_{n+1}), \mathrm{d})$ *restricts to the Heckenberger–Kolb calculus on the quantum Grassmannians, that is, the A-series quantum flag manifolds of Hermitian symmetric type.*

Corollary

For both I and I' the differential calculus $(\Omega_q^\bullet(F_{n+1}), \mathrm{d})$ restricts to the Heckenberger–Kolb calculus on the quantum Grassmannians, that is, the A-series quantum flag manifolds of Hermitian symmetric type.

Corollary

For all other A-series quantum flag manifolds $\mathcal{O}_q(G/L_S)$, for both I and I' the differential calculus $(\Omega_q^\bullet(F_{n+1}), \mathrm{d})$ restricts to a differential calculus on $\mathcal{O}_q(G/L_S)$ of classical dimension.

5: Noncommutative Kähler Structures

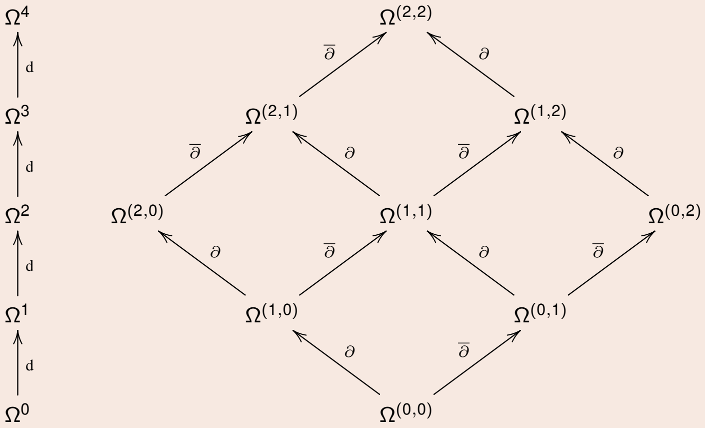
Definition

An *almost complex structure* for a differential $*$-calculus $\Omega^\bullet$, is an $\mathbb{N}_0^2$-algebra grading $\bigoplus_{(a,b)\in\mathbb{N}_0^2}\Omega^{(a,b)}$ for $\Omega^\bullet$ such that, for all $(a,b)\in\mathbb{N}_0^2$:

1. $\Omega^k = \bigoplus_{a+b=k}\Omega^{(a,b)}$;
2. $*(\Omega^{(a,b)}) = \Omega^{(b,a)}$,
3. $\mathrm{d}\Omega^{(a,b)} \subseteq \Omega^{(a+1,b)} \oplus \Omega^{(a,b+1)}$.

Example

Consider the example of 4-dimensional complex manifold:



Definition

Defining two operators $\partial, \overline{\partial} : \Omega^\bullet \to \Omega^\bullet$ by

$$\partial|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a+1,b)}} \circ \mathrm{d}, \qquad \overline{\partial}|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a,b+1)}} \circ \mathrm{d},$$

Definition

Defining two operators $\partial, \overline{\partial} : \Omega^\bullet \to \Omega^\bullet$ by

$$\partial|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a+1,b)}} \circ \mathrm{d}, \qquad \overline{\partial}|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a,b+1)}} \circ \mathrm{d},$$

condition 3 holds if and only if

$$\mathrm{d} = \partial + \overline{\partial}.$$

Definition

Defining two operators $\partial, \overline{\partial} : \Omega^\bullet \to \Omega^\bullet$ by

$$\partial|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a+1,b)}} \circ \mathrm{d}, \qquad \overline{\partial}|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a,b+1)}} \circ \mathrm{d},$$

condition 3 holds if and only if

$$\mathrm{d} = \partial + \overline{\partial}.$$

Theorem (Newlander–Nirenberg 1957)

For a smooth manifold M, an $\mathbb{N}_0^2$-grading of $\Omega^\bullet(M)$, satisfying conditions 1 and 2, comes from a holomorphic atlas on M if and only if $\mathrm{d} = \partial + \overline{\partial}$.

Definition (R.Ó B. '17)

An **Hermitian structure** for a differential calculus of total dimension $2n$ is a pair $(\Omega^{(\bullet,\bullet)}, \sigma)$, where

1. $\Omega^{(\bullet,\bullet)}$ is complex structure for $\Omega^\bullet$,
2. $\sigma \in \Omega^{(1,1)}$ is a central real ($\sigma^* = \sigma$) form ,
3. isomorphisms are given by

$$L^{n-k} : \Omega^k \to \Omega^{2n-k}, \qquad \omega \mapsto \sigma^{n-k} \wedge \omega,$$

for all $1 \leq k < n$.

Definition (R.Ó B. '17)

An **Hermitian structure** for a differential calculus of total dimension $2n$ is a pair $(\Omega^{(\bullet,\bullet)}, \sigma)$, where

1. $\Omega^{(\bullet,\bullet)}$ is complex structure for $\Omega^\bullet$,
2. $\sigma \in \Omega^{(1,1)}$ is a central real ($\sigma^* = \sigma$) form ,
3. isomorphisms are given by

$$L^{n-k} : \Omega^k \to \Omega^{2n-k}, \qquad \omega \mapsto \sigma^{n-k} \wedge \omega,$$

for all $1 \leq k < n$.

Definition (R. Ó B. '17)

A **Kähler structure** is an Hermitian structure $(\Omega^{(\bullet,\bullet)}, \kappa)$ such that $\mathrm{d}\kappa = 0$.

- The $(1,1)$-form in the above definition generalises the properties of the fundamental form of an Hermitian/Kähler metric

$$\text{Hermitian metric } g \quad \Rightarrow \quad \text{fundamental form } \sigma := g(-, I(-)).$$

- In the noncommmutative setting it makes more sense to reverse this order of construction

$$\text{Hermitian metric } g \quad \Leftarrow \qquad \text{fundamental form } \sigma.$$

6: A Kähler Structure for the *A*-Series Quantum Flag Manifolds

Theorem (RÓB, P. Somberg 2021)

For each A-series quantum flag manifold $\mathcal{O}_q(SU_{n+1}/L_S)$, and either choice of longest Weyl group element decomposition, the differential calculus admits a right $U_q(\mathfrak{sl}_{n+1})$-covariant complex structure, corresponding to the decomposition of the tangent space

$$T = T^{(1,0)} \oplus T^{(0,1)} := \operatorname{span}\{F_{\beta_i}\} \oplus \operatorname{span}\{E_{\beta_i}\}.$$

6: A Kähler Structure for the *A*-Series Quantum Flag Manifolds

Theorem (RÓB, P. Somberg 2021)

For each A-series quantum flag manifold $\mathcal{O}_q(SU_{n+1}/L_S)$, and either choice of longest Weyl group element decomposition, the differential calculus admits a right $U_q(\mathfrak{sl}_{n+1})$-covariant complex structure, corresponding to the decomposition of the tangent space

$$T = T^{(1,0)} \oplus T^{(0,1)} := \mathrm{span}\{F_{\beta_i}\} \oplus \mathrm{span}\{E_{\beta_i}\}.$$

Theorem (RÓB, P. Somberg 2021)

Each complex structure admits a right $U_q(\mathfrak{sl}_{n+1}))$-covariant Kähler structure, with positive definite metric.

7: Hilbert Space Completions

- What about the analysis? In particular, what about the connection with Connes' theory of spectral triples?

7: Hilbert Space Completions

- What about the analysis? In particular, what about the connection with Connes' theory of spectral triples?
- Composing g with the Haar state of $\mathcal{O}_q(G)$, we get an inner product, and a Hilbert space completion $L^2(\Omega^\bullet(G/L_S))$.

7: Hilbert Space Completions

- What about the analysis? In particular, what about the connection with Connes' theory of spectral triples?
- Composing g with the Haar state of $\mathcal{O}_q(G)$, we get an inner product, and a Hilbert space completion $L^2(\Omega^\bullet(G/L_S))$.
- (In fact, we also get a Hilbert C^*-module, but that's another story.)
- The Dolbeault–Dirac operator

$$D_{\overline{\partial}} := \overline{\partial} + \overline{\partial}^*$$

is a densely-defined and essentially self-adjoint.

7: Hilbert Space Completions

- What about the analysis? In particular, what about the connection with Connes' theory of spectral triples?
- Composing g with the Haar state of $\mathcal{O}_q(G)$, we get an inner product, and a Hilbert space completion $L^2(\Omega^\bullet(G/L_S))$.
- (In fact, we also get a Hilbert C^*-module, but that's another story.)
- The Dolbeault–Dirac operator

$$D_{\overline{\partial}} := \overline{\partial} + \overline{\partial}^*$$

is a densely-defined and essentially self-adjoint.
- We have a faithful $*$-algebra representation

$$\rho : \mathcal{O}_q(G/L_S) \to \mathbb{B}(L^2(\Omega^\bullet)).$$

To have a spectral triple, we also need

1. $[D_{\overline{\partial}}, b]$ is bounded, for all $b \in \mathcal{O}_q(G/L_S)$,
2. $(D_{\overline{\partial}} - i)^{-1}$ is a compact operator.

Theorem (B. Das, R. Ó B., P. Somberg '20)

A Dolbeault–Dirac pair of spectral triples is given by

$$\Big(\mathcal{O}_q(\mathbb{CP}^n), L^2\big(\Omega^{(\bullet,0)}\big), D_{\partial}\Big), \qquad \Big(\mathcal{O}_q(\mathbb{CP}^n), L^2\big(\Omega^{(0,\bullet)}\big), D_{\overline{\partial}}\Big).$$

Moreover,

$$\mathrm{index}(D_{\overline{\partial}}) = \sum_{k=0}^{n} \dim(H^{(0,k)}) = \dim(H^{(0,0)}) = 1,$$

meaning the associated K-homology class is non-trivial.

Theorem (F. Díaz-Garcia, R. Ó B., E. Wagner '21)

A Dolbeault–Dirac pair of spectral triples is given by

$$\Big(\mathcal{O}_q(\mathbf{Q}_5), L^2\big(\Omega^{(\bullet,0)}\big), D_{\partial}\Big), \qquad \Big(\mathcal{O}_q(\mathbf{Q}_5), L^2\big(\Omega^{(0,\bullet)}\big), D_{\overline{\partial}}\Big).$$

Moreover,

$$\mathrm{index}(D_{\overline{\partial}}) = \sum_{k=0}^{n} \dim(H^{(0,k)}) = \dim(H^{(0,0)}) = 1,$$

meaning the associated K-homology class is non-trivial.

Conjecture

The A-series construction of a q-deformed de Rham complex extends to a general $U_q(\mathfrak{g})$-construction, with all the associated noncommutative Kähler geometry.

Moreover, the Dolbeault–Dirac operator has compact resolvent.

Conjecture

The A-series construction of a q-deformed de Rham complex extends to a general $U_q(\mathfrak{g})$-construction, with all the associated noncommutative Kähler geometry.

Moreover, the Dolbeault–Dirac operator has compact resolvent.

- As support of this conjecture, we know that twisting $D_{\overline{\partial}}$ by a negative line bundle gives a Fredholm operator.

Conjecture

The A-series construction of a q-deformed de Rham complex extends to a general $U_q(\mathfrak{g})$-construction, with all the associated noncommutative Kähler geometry.

Moreover, the Dolbeault–Dirac operator has compact resolvent.

- As support of this conjecture, we know that twisting $D_{\overline{\partial}}$ by a negative line bundle gives a Fredholm operator.
- However, $[D_{\overline{\partial}}, b]$ is not in general bounded . . . so we may need to generalise to twisted spectral triples . . .